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Heat transfer through periodic macro-contact with constriction

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Abstract

A quadrupole method is developed to solve heat transfer through a periodic macro-contact with time varying constriction. The solution is based on Fourier developments of time periodic variables. The result shows that it is necessary to introduce the concept of "building-up" of constriction to explain the thermal behavior for short and moderate periods. It is demonstrated that three characteristic times govern the problem: contact period, characteristic time of the rod and "building-up" time of constriction. Simplified schemes of the apparent resistance are presented corresponding to three limiting states. The analytical approach is validated by a numerical solution. © 2002 Elsevier Science Ltd. All rights reserved.

1. Introduction

The imperfection of contact surfaces yields a constriction of heat flow through the interface. This phenomenon constitutes an important subject of study in the field of heat transfer across solid surfaces permanently held in contact [1–5]. Nevertheless, very little work is available in the literature on the investigation of thermal constriction in periodically contacting regions, despite the importance of this problem in numerous practical applications.

Conductive heat transfer through periodically contacting surfaces has been modeled in both theoretical [6–13] and experimental [14–17] works. They correspond to the assumption of a uniform contact conductance at the whole interface, and thus to a one-dimensional heat flow in contacting regions. However, a time-varying thermal constriction of the heat flow appears because the actual contact area is exceedingly small compared with the apparent contact area in practical cases. It is thus necessary to consider the influence of constriction on heat transfer through periodically contacting surfaces.

The valve-seat periodic contact in an internal combustion engine can be considered as a typical example of its industrial applications [18]. As it is shown in various studies, both theoretical and experimental, the main problem on this topic lies on the dependence of the apparent contact resistance (the time average during a period) on the period. We will be engaged to analyze theoretically this dependence in the presence of a macro-constriction of heat flux lines and demonstrate the importance of the "building-up" time of this macro-constriction.

Toward this end, we develop a quadrupole model [19], which contains a term associated to the thermal constriction, in order to solve conductive transfer through a periodic macro-contact. This model is used to discuss the influence of thermal constriction on the apparent system resistance in a wide range of contact frequency. We analyze moreover the characteristic time of "building-up" of the constriction in a simple case and its influence on the apparent resistance of periodic contact. Consequently, simplified schemes of the apparent system resistance are obtained for the three limiting states.

The quadrupole model is validated by a finite difference solution.

2. Mathematical model

We consider a cylindrical rod of length l with a uniform cross-section of radius R (heat conductivity k,

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Nomenclature

A, B, C, D	quadrupole coefficients
a	heat diffusivity
h	periodic contact conductance at the interface
J_0, J_1	Bessel functions of order 0 and 1
k	heat conductivity
l	rod length
p	Laplace variable
Q	heat flux
q	heat flux density
R	rod radius
r	radial coordinate
r_0	asperity radius
r_{app}	apparent system resistance
r_{b}	thermal resistance of the rod
r_{c}	contact resistance
r_{ct}	thermal constriction resistance
T	temperature
T_0	imposed fixed end temperature of the rod
T_{c}	imposed contact surface temperature
t	time variable
x	axial coordinate
z_{ct}	constriction impedance
z_n	term associated with the thermal constriction contained in the quadrupole

Greek symbols

α_n	nth root of the equation $J_1(\alpha R) = 0$
$\delta(t)$	Dirac function
Φ	Laplace transform of heat flux
γ_n	$\gamma_n^2 = \alpha_n^2 + \mathrm{i}\omega_n/a$
θ	Laplace transform of temperature
τ	contact period
τ_{b}	characteristic time of the rod
τ_{ct}	characteristic time of the constriction
ω_n	$n\omega$ (ω is the pulsation)

Subscripts

0	at $x = 0$
$0, m, \infty$	three limiting states, corresponding to short, moderate and long periods, respectively
1, 2	related to phases 1, 2 (contact and noncontact)
a	related to the asperity
l	at $x = l$
n	related to the nth term of Fourier development

Superscripts

$-$	average quantity
$*$	dimensionless quantity

heat diffusivity a), which is insulated laterally so that no heat transfer takes place from the sides. One end ($x = 0$) is held at a uniform temperature T_0 while the other end ($x = l$) is brought into periodic contact (contact–noncontact) with a plane kept at constant temperature T_{c}. This contact involves a disk (the asperity) of radius $r_0 < R$ (Fig. 1). We assume that: the contact conductance is time dependent and uniform over the whole asperity (actual contact area); there is no heat transfer through the remaining area surrounding the asperity; the thickness of the asperity is neglected; and the periodic condition is established.

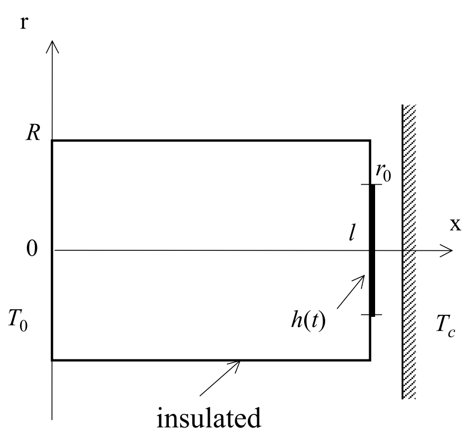

Fig. 1. Scheme of macro-contact.

The governing heat conduction problem is therefore defined as

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\partial^2 T}{\partial x^2} = \frac{1}{a}\frac{\partial T}{\partial t}, \tag{1}$$

$$T = T_0 \quad \text{for } x = 0, \tag{1a}$$

$$-k\frac{\partial T}{\partial x} = \begin{cases} h(t)(T - T_{\mathrm{c}}) & \text{for } 0 < r < r_0, \\ 0 & \text{for } r_0 < r < R, \end{cases} \quad x = l, \tag{1b}$$

$$\frac{\partial T}{\partial r} = 0 \quad \text{for } r = R, \tag{1c}$$

where the contact resistance per unit area $1/h(t)$ varies periodically with time in a period $\tau = \tau_1 + \tau_2$. It is equal to $1/h_1$ during the contact phase τ_1 and to $1/h_2$ during the noncontact phase τ_2.

3. Resolution of the problem by the quadrupole method

The quadrupole method has been used to solve transient conductive transfer problems using inverse transfer matrices that link Laplace transforms of input and output temperatures and heat fluxes [19–21].

Degiovanni et al. [22] presented an approximate analytical model of thermal constriction in transient state. Using the average temperatures and fluxes (in the form of Laplace transform), they gave a quadrupole representation of the transient constriction. We extend here the same approach to periodic macro-contact with transient constriction.

3.1. Thermal quadrupole related to a constriction

One reviews here in brief the general problem of a constriction in variable state (Fig. 1)

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{\partial^2 T}{\partial x^2}=\frac{1}{a}\frac{\partial T}{\partial t}, \tag{2}$$

$$-k\frac{\partial T}{\partial x}=\begin{cases} q(r,t) & \text{for } 0<r<r_0, \\ 0 & \text{for } r_0<r<R, \end{cases} \quad x=l, \tag{2a}$$

$$\frac{\partial T}{\partial r}=0 \quad \text{for } r=R, \tag{2b}$$

any condition independent of r for $x=0$, (2c)

$$T=0 \quad \text{at } t=0. \tag{2d}$$

Use of the Laplace transform with the initial condition (2d) leads to

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \theta}{\partial r}\right)+\frac{\partial^2 \theta}{\partial x^2}=\frac{p}{a}\theta, \tag{3}$$

$$-k\frac{\partial \theta}{\partial x}=\begin{cases} \varphi(r,p) & \text{for } 0<r<r_0, \\ 0 & \text{for } r_0<r<R, \end{cases} \quad x=l, \tag{3a}$$

$$\frac{\partial \theta}{\partial r}=0 \quad \text{for } r=R. \tag{3b}$$

When $l>R$, which is always the case in practice, the quadrupole associated with problem (3), is independent of the boundary condition in $x=0$, i.e.

$$\begin{bmatrix} \bar{\theta}_0 \\ \Phi_0 \end{bmatrix}=\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} \bar{\theta}_a \\ \Phi_a \end{bmatrix}, \tag{4}$$

where $\bar{\theta}_0$ denotes the Laplace transform of the average temperature over the surface at $x=0$, and $\bar{\theta}_a$ the average temperature over the asperity area $0<r<r_0$ at $x=l$; Φ_0 and Φ_a are the Laplace heat fluxes at $x=0$ and $x=l$.

3.2. Quadrupoles in established periodic state

Assuming that periodic state is established, the temperature and the flux density varying periodically with time can be developed in Fourier series

$$T(x,r,t)=\sum_{n=-\infty}^{n=+\infty}\left[T_n(x,r)e^{i\omega_n t}\right], \tag{5a}$$

$$q(x,r,t)=\sum_{n=-\infty}^{n=+\infty}\left[q_n(x,r)e^{i\omega_n t}\right], \tag{5b}$$

where $\omega_n=n\omega=2\pi n/\tau$.From problem (2), to be solved, we have then the following equation in T_n:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_n}{\partial r}\right)+\frac{\partial^2 T_n}{\partial x^2}=\frac{i\omega_n}{a}T_n, \tag{6}$$

$$-k\frac{\partial T_n}{\partial x}=\begin{cases} q_n(r) & \text{for } 0<r<r_0, \\ 0 & \text{for } r_0<r<R, \end{cases} \quad x=l, \tag{6a}$$

$$\frac{\partial T_n}{\partial r}=0 \quad \text{for } r=R. \tag{6b}$$

Comparing system (6) with Eq. (3), it is obvious that the solution of (6) can be put in quadrupole form as below:

$$\begin{bmatrix} \bar{T}_{0,n} \\ Q_{0,n} \end{bmatrix}=\begin{bmatrix} A_n & B_n \\ C_n & D_n \end{bmatrix}\begin{bmatrix} \bar{T}_{a,n} \\ Q_{a,n} \end{bmatrix}, \tag{7}$$

where $\bar{T}_{0,n}$, $\bar{T}_{a,n}$, $Q_{0,n}$ and $Q_{a,n}$ denote, respectively, the nth terms of Fourier developments of the average temperatures and the total flux over the whole surface at $x=0$ and the asperity area $0<r<r_0$ at $x=l$, that is to say

$$\begin{aligned} \bar{T}_0&=\frac{2}{R^2}\int_0^R T_{x=0}r\,dr, \\ \bar{T}_a&=\frac{2}{r_0^2}\int_0^{r_0} T_{x=l}r\,dr, \\ Q_0&=2\pi\int_0^R\left(-k\frac{\partial T}{\partial x}\right)_{x=0}r\,dr, \\ Q_a&=2\pi\int_0^{r_0}\left(-k\frac{\partial T}{\partial x}\right)_{x=l}r\,dr. \end{aligned} \tag{8}$$

Substituting p by $i\omega_n$ in the expression (4), the quadrupole is written as

$$\begin{bmatrix} A_n & B_n \\ C_n & D_n \end{bmatrix}=\begin{bmatrix} A'_n & B'_n \\ C'_n & D'_n \end{bmatrix}\begin{bmatrix} 1 & z_n \\ 0 & 1 \end{bmatrix}, \tag{9}$$

where the coefficients of the first sub-quadrupole related to the unperturbed medium are given below:

$$A'_n=D'_n=\cosh(\sqrt{i\omega_n/a}\,l), \tag{9a}$$

$$C'_n=\pi R^2 k\sqrt{i\omega_n/a}\sinh(\sqrt{i\omega_n/a}\,l), \tag{9b}$$

$$B'_n=\frac{\sinh\left(\sqrt{i\omega_n/a}\,l\right)}{\pi R^2 k\sqrt{i\omega_n/a}}, \tag{9c}$$

$$\left(\text{for } n=0,\ A'_0=D'_0=1,\ C'_0=0 \text{ and } B'_0=r_b=\frac{l}{\pi R^2 k}\right),$$

and the term associated with the constriction of the heat flux lines

$$z_n = \sum_{m=1}^{\infty} \frac{2J_1(\alpha_m r_0)\int_0^{r_0} rq_{l,n}(r)J_0(\alpha_m r)\mathrm{d}r}{\pi k\alpha_m r_0 R^2 \gamma_m J_0^2(\alpha_m R)\int_0^{r_0} rq_{l,n}(r)\mathrm{d}r}, \tag{10}$$

where α_m stands for the roots of $J_1(\alpha R) = 0$, and $\gamma_m^2 = \alpha_m^2 + \mathrm{i}\omega_n/a$.

Notes.

- For the periodic case, the inputs and the outputs of the transfer matrix are, respectively, the coefficients of the Fourier developments of the average temperatures and of the total flux over the surface at $x = 0$ and over the asperity area at $x = l$.
- This model takes into account the "building-up" of the thermal constriction through the z_n term.
- This model makes sense only under the condition $l > R$ [22], which means that the constriction is established axially, i.e. the transfer is one-directional at $x = 0$.

3.3. Solution for periodic macro-contact

Using the average temperatures, we have obtained an approximate analytical model for the periodic conduction with constriction. It presents the same quadrupole representation as that of one-dimensional conduction. We follow identical steps, which have been used in one-dimensional case of periodic contact [23], to solve this model for the case of periodic macro-contact. In fact, the only additional difficulty lies in the calculation of the constriction term z_n.

To connect the flux at the interface to the periodic contact conductance $h(t)$, boundary condition (1b) is rewritten using average values over the asperity $(0 < r < r_0)$

$$Q_\mathrm{a} = \pi r_0^2 h(t)(\bar{T}_\mathrm{a} - T_\mathrm{c}). \tag{11}$$

The periodic contact conductance $h(t)$ is developed into Fourier series as well

$$h(t) = \sum_{n=-\infty}^{n=\infty} h_n \mathrm{e}^{\mathrm{i}\omega_n t}. \tag{12}$$

Then condition (11) becomes

$$\sum_{n=-\infty}^{n=+\infty} Q_{\mathrm{a},n}\mathrm{e}^{\mathrm{i}\omega_n t} = \pi r_0^2 \left(\sum_{n=-\infty}^{n=+\infty}\left[h_n \mathrm{e}^{\mathrm{i}\omega_n t}\right]\right) \times \left(\sum_{n=-\infty}^{n=+\infty}\left[(\overline{T}_{\mathrm{a},n} - T_{\mathrm{c},n})\mathrm{e}^{\mathrm{i}\omega_n t}\right]\right). \tag{13}$$

From this expression and quadrupole equation (7), coupled equations are easily obtained

$$Q_{\mathrm{a},n} = \pi r_0^2 \sum_{m=-\infty}^{m=+\infty}\left[h_{n-m}(\overline{T}_{\mathrm{a},m} - T_{\mathrm{c},m})\right], \tag{14a}$$

$$\overline{T}_{0,n} = A_n \overline{T}_{\mathrm{a},n} + B_n Q_{\mathrm{a},n}, \tag{14b}$$

and therefore $\bar{T}_{\mathrm{a},n}$ is the solution of the following equation:

$$\pi r_0^2 \sum_{m=-\infty}^{m=+\infty}\left[h_{n-m}(\overline{T}_{\mathrm{a},m} - T_{\mathrm{c},m})\right] + \frac{A_n}{B_n}\overline{T}_{\mathrm{a},n} = \frac{\overline{T}_{0,n}}{B_n}. \tag{15}$$

The infinite summation is truncated so that only the terms corresponding to $-N \leqslant m \leqslant N$ are kept in the practical calculation.

The following vectors are defined to facilitate the calculation:

$$\mathbf{T_l} = [T_{l,n}], \quad \mathbf{T_c} = [T_{\mathrm{c},n}], \quad \mathbf{T_0} = [T_{0,n}] \quad \text{and } \mathbf{H} = [h_{n+N}]. \tag{16}$$

Eq. (15) can therefore be written as a matrix equation of finite size:

$$\left(\mathbf{toeplitz}(\mathbf{H},\mathbf{H}') + \mathbf{diag}\left[\frac{A_n}{B_n}\right]\right)\mathbf{T_l} = \left[\frac{T_{0,n}}{B_n}\right] + \mathbf{toeplitz}(\mathbf{H},\mathbf{H}')\mathbf{T_c}, \tag{17}$$

where $\mathbf{diag}[A_n/B_n]$ is a diagonal matrix and $\mathbf{toeplitz}(\mathbf{H},\mathbf{H}')$ is an asymmetrical Toeplitz matrix, the first row of which is $\mathbf{H}$ and the first line $\mathbf{H}'$

$$\mathbf{toeplitz}(\mathbf{H},\mathbf{H}') = \begin{bmatrix} h_0 & h_{-1} & \cdots & h_{-2N} \\ h_1 & h_0 & \cdots & h_{-2N+1} \\ \vdots & \vdots & \ddots & \vdots \\ h_{2N} & h_{2N-1} & \cdots & h_0 \end{bmatrix}. \tag{18}$$

Matrix solvers of MATLAB type [24] allow a very simple calculation of this kind of expression.

Appendix A gives the Fourier development of the contact conductance $h(t)$ in the studied case of an intermittent contact (contact-noncontact). In fact, thanks to the use of Fourier development of periodic quantities, the above method is valid for contacts with any periodic boundary contact condition at the interface.

In order to calculate z_n, a usual approximation lies in assuming a uniform flux density on the actual contact area of radius r_0. Thus expression (10) is reduced to

$$z_n = \sum_{m=1}^{\infty} \frac{4J_1^2(\alpha_m r_0)}{\pi k\alpha_m^2 r_0^2 R^2 \gamma_m J_0^2(\alpha_m R)}, \tag{19}$$

where the zeros $(j_{v,m})$ of Bessel functions are approximated by [25]

$$j_{v,m} \approx \beta - \frac{\mu - 1}{8\beta} - \frac{4(\mu-1)(7\mu-31)}{3(8\beta)^2} - \frac{32(\mu-1)(83\mu^2 - 982\mu + 3779)}{15(8\beta)^3} - \frac{64(\mu-1)(6949\mu^3 - 153855\mu^2 + 1585743\mu - 6277237)}{105(8\beta)^7} - \cdots$$

with $\mu = 4v^2$ and $\beta = (m + \frac{1}{2}v - \frac{1}{4})\pi$.

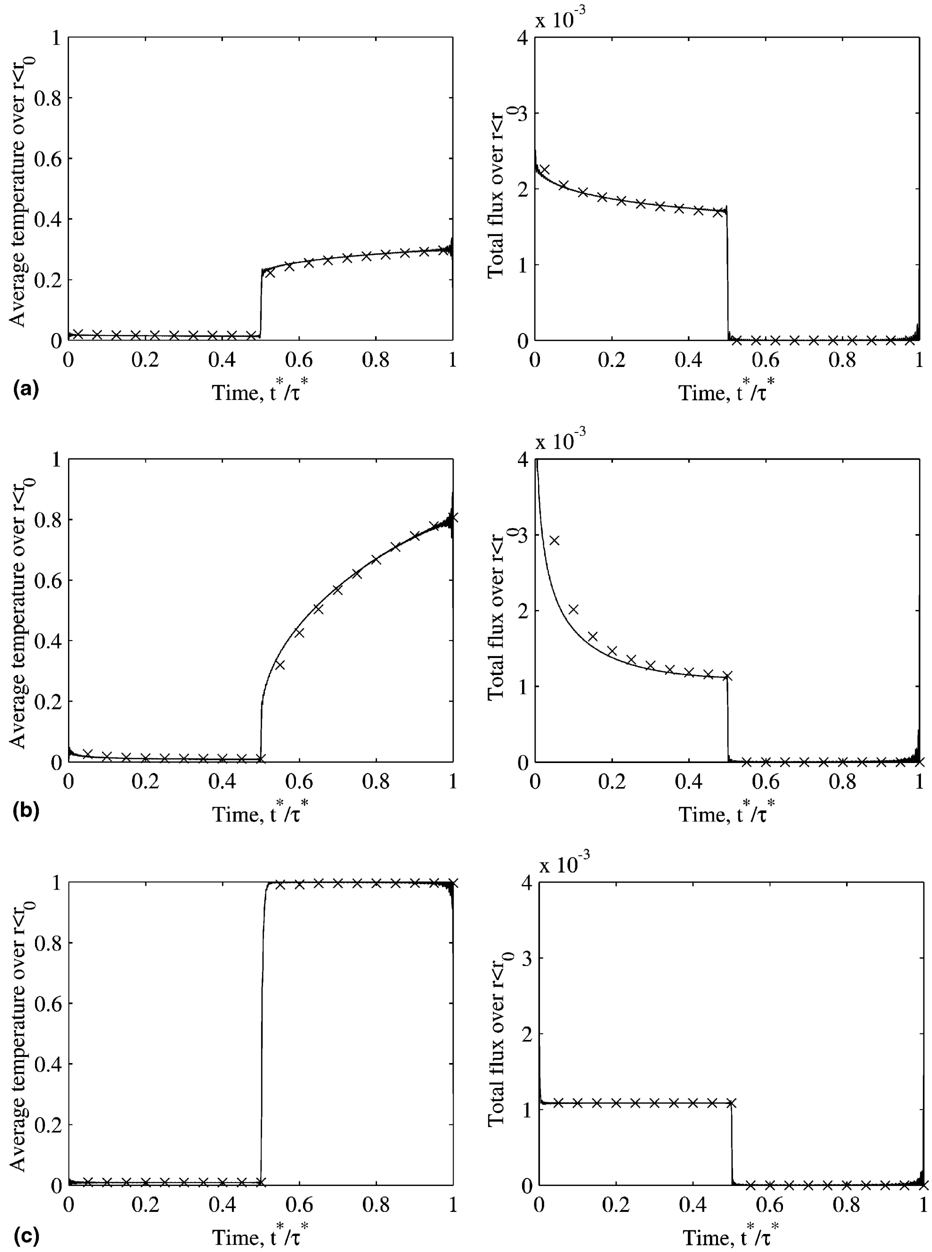

Fig. 2. Variations of average temperature and total flux during a period ($r_0/R = 0.10, R/l = 0.02$). ($\times$) Finite difference; (-) quadrupole. (a) $\tau^* = 10^{-2}$; (b) $\tau^* = 10^0$; (c) $\tau^* = 10^2$.

Notes.

- When the pulsation $\omega_n = 2\pi n/\tau$ approaches zero, the constriction term z_n becomes the constriction resistance in steady state. An approximate formula is available [26]

$$z_0 = \frac{A_0}{\pi k r_0} \tag{20}$$

with

$$A_0 = 0.848\text{–}1.093\frac{r_0}{R} + 0.245\left(\frac{r_0}{R}\right)^{3.75}$$

(when $r_0/R \to 0$, $A_0 \to 8/3\pi = 0.848$).

- z_n is a decreasing function of the pulsation.
- In the case $R \gg r_0$, the constriction term is identical to its value obtained for a semi-infinite domain [22], i.e.

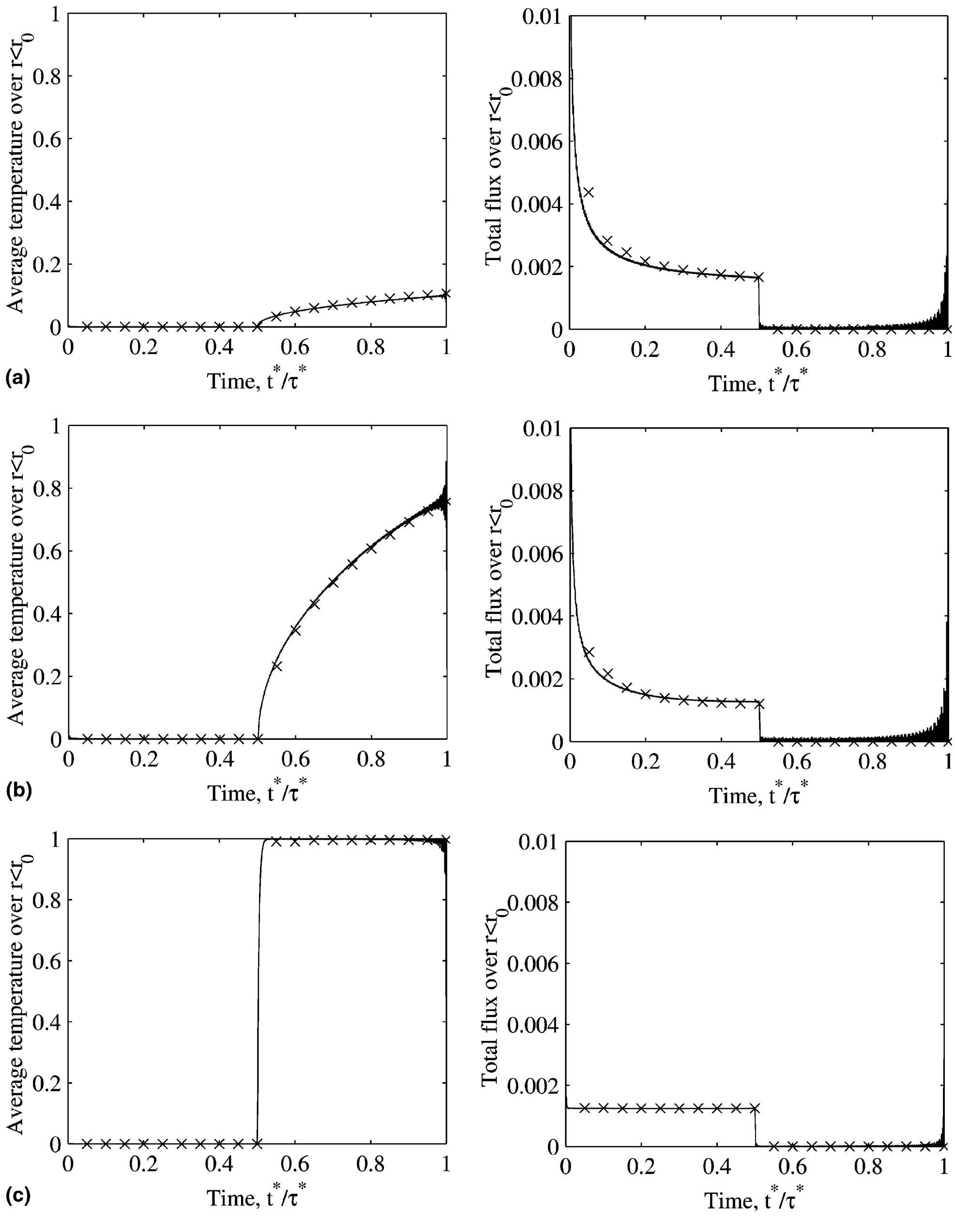

Fig. 3. Variations of average temperature and total flux during a period ($r_0/R = 0.80, R/l = 0.02$). (×) Finite difference; (-) quadrupole. (a) $\tau^* = 10^{-2}$; (b) $\tau^* = 10^0$; (c) $\tau^* = 10^2$.

$$z_n' = \frac{2}{\pi k r_0} \int_0^\infty \frac{J_1^2(\varepsilon)}{\varepsilon} \frac{1}{\sqrt{\varepsilon^2 + \mathrm{i}\omega_n r_0^2/a}} \mathrm{d}\varepsilon. \tag{21}$$

An approximate formula of this integral is given [27]

$$z_n' \approx \frac{8}{3\pi^2 k r_0 \left(1 + \frac{8}{3\pi}\sqrt{\mathrm{i}\omega_n r_0^2/a}\right)}. \tag{22}$$

This approximation is obtained by setting in parallel the constriction resistance, here $z_0' = 8/(3\pi^2 k r_0)$, with the impedance of a semi-infinite cylinder of radius r_0, i.e. $1/\sqrt{k\rho c}\pi r_0^2\sqrt{\mathrm{i}\omega_n}$. We get this approximation from the fact that the formula (21) tends to the impedance of a semi-infinite cylinder of radius r_0 while ω_n tending to infinity. In fact when $\omega_n \to \infty \quad z'_n = \frac{2}{\pi k r_0} \frac{1}{\sqrt{\mathrm{i}\omega_n r_0^2/a}} \int_0^\infty \frac{J_1^2(\varepsilon)}{\varepsilon} \mathrm{d}\varepsilon,$

and

$$\int_0^\infty \frac{J_1^2(\varepsilon)}{\varepsilon} \mathrm{d}\varepsilon = \frac{1}{2}.$$

(This result shows that the transfer is uni-directional for short times.)

- Using the same approach with the formula (19), an approximation is obtained

$$z_n \approx \frac{A_0}{\pi k r_0 (1 + A_0 \sqrt{\mathrm{i}\omega_n r_0^2/a})}. \tag{23}$$

4. Results and discussion

For the macro-contact in periodic state, the apparent system resistance r_{app} is defined as

$$r_{\mathrm{app}} = \frac{\Delta T}{\bar{Q}} \quad \text{with } \Delta T = \bar{T}_0 - T_{\mathrm{c}}, \tag{24}$$

where $\bar{T}_0$ and $\bar{Q}$ are time averages of the temperature and the heat flux going through the system ($\bar{Q} = \bar{Q}_{\mathrm{a}}$), respectively.

While presenting the results of the calculation, we pay a particular attention to the apparent resistance varying with the contact period in the case of an intermittent contact ($h_1 \approx \infty$, $h_2 \approx 0$) and $\tau_1 = \tau_2$. All the results are given in dimensionless form. So the following dimensionless quantities are introduced: space variables $x^* = x/l$ and $r^* = r/l$, time variable $t^* = t/\tau_{\mathrm{b}}$ and temperature $T^* = (T - T_{\mathrm{c}})/(T_0 - T_{\mathrm{c}})$; period $\tau^* = \tau/\tau_{\mathrm{b}}$; contact conductance $h^* = h\pi R^2 r_{\mathrm{b}}$ and apparent system resistance $r^*_{\mathrm{app}} = r_{\mathrm{app}}/r_{\mathrm{b}}$, where the thermal resistance of the rod is $r_{\mathrm{b}} = l/\pi R^2 k$ and its thermal characteristic time $\tau_{\mathrm{b}} = l^2/a$.

4.1. Verification of the quadrupole model

The quadrupole model is based on averaged quantities for inputs and outputs. Consequently, the obtained results are the average temperature and the total flux over the asperity area. In order to validate the model and the approximations, we compare the results of the quadrupole method with a numerical solution of the problem [23,28]. The numerical resolution of system (1), is performed by the finite difference method of implicit scheme, where a non-uniform grid is generated in axisymmetric coordinates and an alternate direction line iteration (ADI) is adopted.

Figs. 2 and 3 show the variations of the average temperature and of the total flux over the asperity area, $\overline{T^*_{\mathrm{a}}}$ and Q^*_{a}, during a period obtained by both methods. The results are presented for different contact periods in the cases of small and large asperities. They show a good agreement between the two solutions. However we observe a harmonic oscillation of the temperature caused by the imperfection of the Fourier development for describing a discontinuous function (Fig. 4).

Fig. 5 shows the variation of the apparent system resistance with the contact period for the two methods. Considering the numerical difficulties of both solutions, the agreement is excellent.

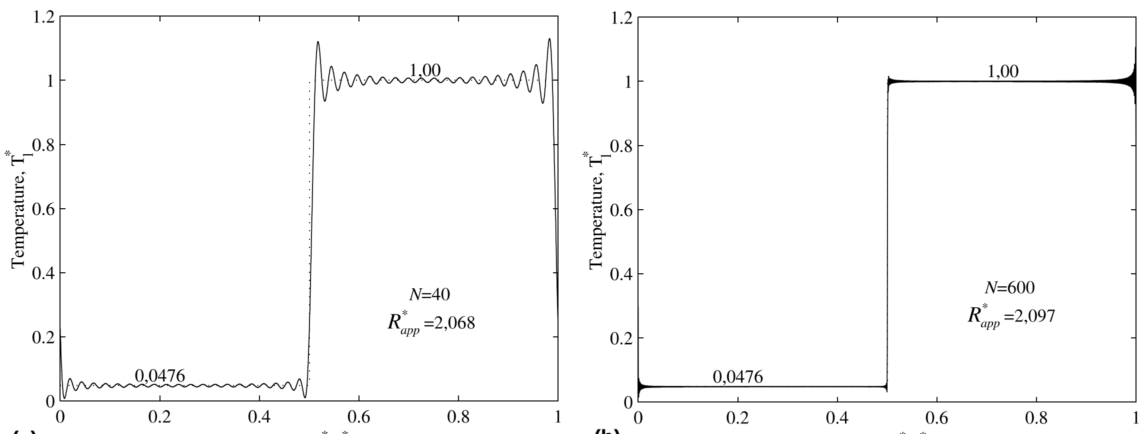

Fig. 4. Convergence of the solution by the quadrupole method ($h_1^* = 20, h_2^* = 0, \tau^* = 10^3$). (-) Result by the quadrupole method; $(\cdots)$ result of steady states. (a) Harmonic oscillation of the temperature T_1^* (for small N); (b) Gibbs phenomenon of the temperature T_1^* (for big N).

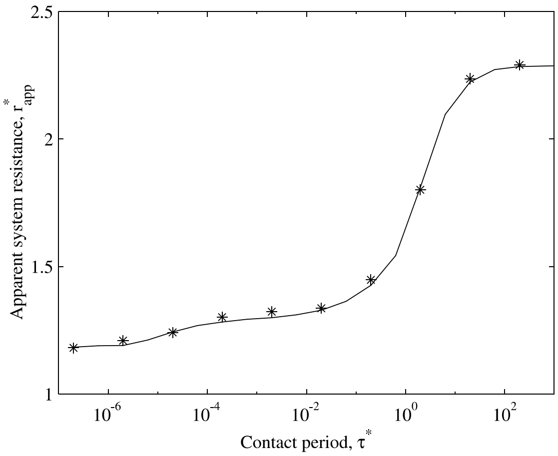

Fig. 5. Comparison of quadrupole result and finite difference solution ($r_0/R = 0.10, R/l = 0.02$). (-) Finite difference; ($*$) quadrupole.

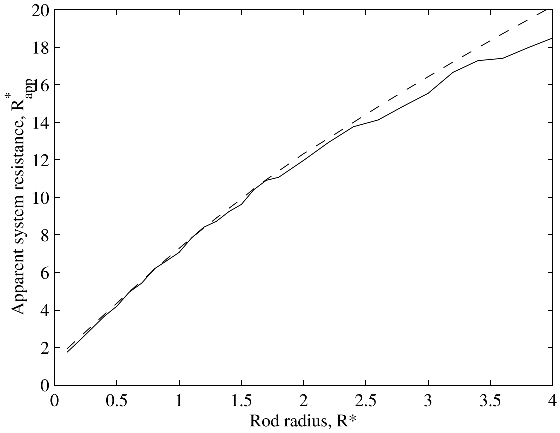

Fig. 6. Apparent system resistance versus R^* ($r_0/R = 0.20$ and $\tau^* = 10^0$). (-) Finite difference; (–) quadrupole.

4.2. Influence of the macro-contact geometry

Degiovanni et al. [22] demonstrated that the quadrupole associated with a thermal constriction is valid under the condition $l > R$ ($\tau^* = 1$ chosen). Fig. 6 shows the variation of the apparent resistance of the system with R/l, while maintaining the r_0/R ratio constant. The preceding condition is thus confirmed. Fig. 7 shows the finite difference solutions of the temperature distribution in the rod at times $\tau/2$ and τ in the cases $R/l = 1.0$ and $R/l = 4.0$. In the case $R/l = 4.0$, the heat flux over the surface $x = 0$ becomes non-uniform (the flux at $r = R$ is 0), which explains the observed deviation of the quadrupole model from the numerical calculation, the quadrupole model being not valid any more.

4.3. Comparison with the results of the one-dimensional model

Fig. 8 presents the apparent resistance of a periodic macro-contact as a function of the contact period as well as the same variation for a one-dimensional periodic contact (a uniform contact conductance over the whole surface at the interface). The one-dimensional result is taken as [23]. The uni-directional case corresponds to the case $r_0 = R$, that is to say the absence of constriction in the rod. In order to compare these two cases, we have reduced the apparent resistance of the uni-directional case to that of the two-directional case for large period state, which amounts to add the constriction resistance to the interface resistance:

$$h_{eq} = \frac{1}{1/h + z_0}. \tag{25}$$

A good agreement between one-dimensional contact and macro-contact models is observed for long periods. The macro-contact curve is characterized by an additional inflexion when compared with the one-dimensional situation. This is the consequence of the building-up of a constriction. We will try to explain this phenomenon by introducing a characteristic time of the ''building-up'' of the constriction.

Some elements can be given for the comparison of the two different calculation techniques. For example, in a case of moderate period ($\tau^* = 1$), it takes approximately 30 min for the finite difference program written in *Fortran77* (30×15 grid, 60 timesteps per with a relative precision lower than 0.1%), and 6 min for the quadrupole program in *Matlab5* (250 terms used in the series) on a Sun-Ultra1/SPARC workstation.

5. ''Building-up'' of thermal constriction

To define a ''characteristic time'' of constriction, we are faced with a simple case of semi-infinite medium

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\partial^2 T}{\partial x^2} = \frac{1}{a}\frac{\partial T}{\partial t}, \tag{26}$$

$$-k\frac{\partial T}{\partial x} = q, \quad 0 < r < r_0, \ x = 0, \tag{26a}$$

$$-k\frac{\partial T}{\partial x} = 0, \quad r > r_0, \ x = 0, \tag{26b}$$

$$T = 0, \quad x \to \infty \text{ and } r \to \infty \text{ and } t = 0. \tag{26c}$$

The analysis is difficult. We present two approaches: the first uses the notion of ''time varying resistance'', the second the notion of ''impedance''.

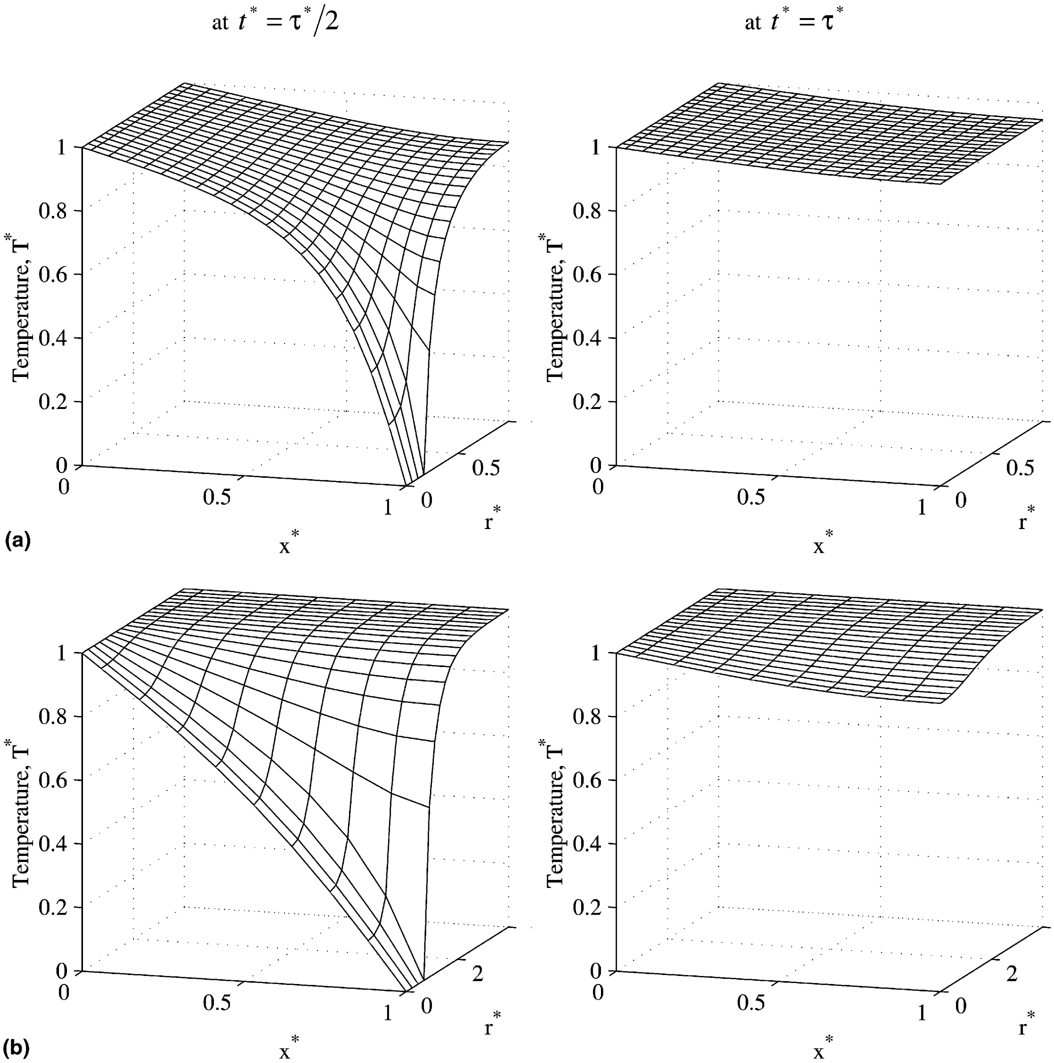

Fig. 7. Temperature distribution in the rod ($\tau^* = 10^0$). (a) $R/l = 1.0,\ r_0/R = 0.2$; (b) $R/l = 4.0,\ r_0/R = 0.2$.

In the first approach, the constriction is characterized at every time by a time varying resistance [29]

$$r_{\mathrm{ct}} = \frac{\frac{2}{r_0^2}\int_0^{r_0} T|_{x=0} r\,\mathrm{d}r}{\pi r_0^2 q}. \tag{27}$$

In the second approach, the constriction is characterized in the Laplace space by its impedance [27]

$$z_{\mathrm{ct}} = \frac{\bar{\theta}}{\Phi}, \tag{28}$$

where $\bar{\theta}$ and Φ are the Laplace transforms of the average temperature and of the heat flux over $0 < r < r_0$ at $x = 0$, respectively.

5.1. "Building-up time" of constriction

5.1.1. Variable constriction "resistance" r_{ct}

We solve the first problem (26) in pulsed regime, i.e. replacing q in (26a) by the Dirac distribution $\delta(t)$

$$-k\frac{\partial T}{\partial x} = \delta(t), \quad 0 < r < r_0, \quad x = 0. \tag{29}$$

The problem is the limit of the problem defined by:

$$-k\frac{\partial T}{\partial x} = 0, \quad x = 0, \tag{30a}$$

$$\left.\begin{array}{ll} \left.\begin{array}{ll} T = \frac{1}{\rho c \mu}, & r < r_0, \\ T = 0, & r > r_0, \end{array}\right\} & x < \mu, \\ T = 0, & x > \mu, \end{array}\right\} \quad t = 0, \tag{30b}$$

where μ denotes an infinitesimal thickness ($\mu \to 0$).

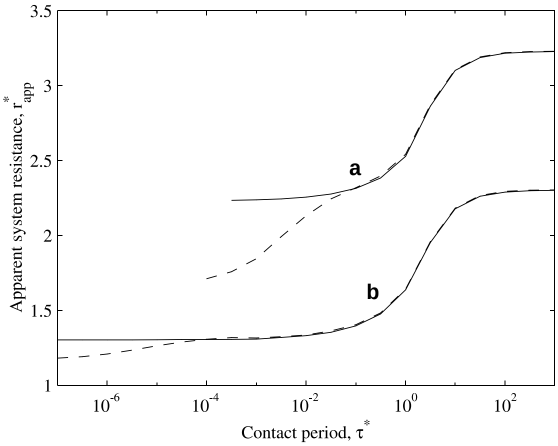

Fig. 8. Comparison of macro-contact with one-dimensional model. (a) $r_0/l = 0.04$, $R/l = 0.20$; (b) $r_0/l = 0.002$, $R/l = 0.020$. (-) One-dimensional model; (–) macro-contact.

The solution of problem (30a) and (30b) by the method of separations of variables is

$$T(r,x,t) = X(x,t)R(r,t) \tag{31}$$

with

$$X(x,t) = \frac{1}{2\mu k/a}\left[\operatorname{erf}\left(\frac{\mu - x}{2\sqrt{\mathrm{a}t}}\right) + \operatorname{erf}\left(\frac{\mu + x}{2\sqrt{\mathrm{a}t}}\right)\right], \tag{32}$$

$$\left(X(x,t) = \frac{1}{k\sqrt{\pi t/a}}\mathrm{e}^{-\frac{x^2}{4\,\mathrm{a}t}},\ \text{as } \mu \to 0\right), \tag{32a}$$

and

$$R(r,t) = \begin{cases} \frac{4}{\pi^2 r_0}\int_0^\infty \mathrm{e}^{-\mathrm{a}t\varepsilon^2}\frac{J_1(r_0\varepsilon)J_0(r\varepsilon)}{U^2\varepsilon^2}\mathrm{d}\varepsilon & \text{for } r < r_0, \\ \frac{2}{\pi}\int_0^\infty \mathrm{e}^{-\mathrm{a}t\varepsilon^2}\frac{J_1(r_0\varepsilon)J_0(r\varepsilon)}{U\varepsilon}\mathrm{d}\varepsilon & \text{for } r > r_0, \end{cases} \tag{33}$$

where $U(\varepsilon) = J_1(r_0\varepsilon)Y_0(r_0\varepsilon) - J_0(r_0\varepsilon)Y_1(r_0\varepsilon)$.

Using Duhamel's theorem, the solution of the problem (26) is obtained

$$T = \int_0^t qX(x,t-\xi)R(r,t-\xi)\mathrm{d}\xi \tag{34}$$

and therefore the temperature for $0 < r < r_0$ at $x = 0$

$$T(r) = \frac{4q}{\pi^2 k\sqrt{\pi r_0^2/a}} \times \int_0^\infty\left(\int_0^t \frac{\mathrm{e}^{-\mathrm{a}\varepsilon^2\tau}}{\sqrt{\tau}}\mathrm{d}\tau\right)\frac{J_0(r\varepsilon)J_1(r_0\varepsilon)}{U^2\varepsilon^2}\mathrm{d}\varepsilon, \quad 0 < r < r_0. \tag{35}$$

Lastly, the constriction "resistance" varying with time is obtained from its definition (27)

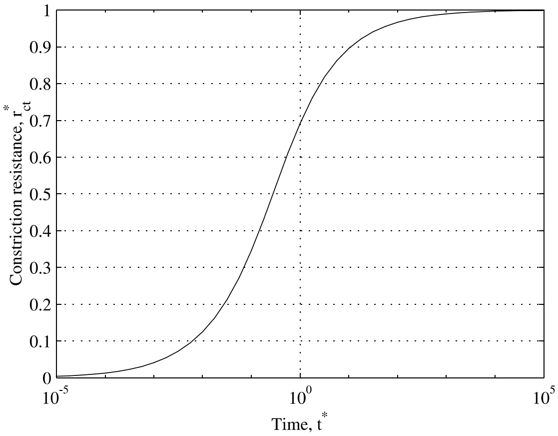

Fig. 9. Constriction resistance versus time.

$$r_{\mathrm{ct}} = \frac{1}{\pi r_0^2}\frac{8}{\pi^2 k r_0^2}\int_0^\infty \operatorname{erf}(\sqrt{\mathrm{a}t}\varepsilon)\frac{J_1^2(r_0\varepsilon)}{U^2(\varepsilon)\varepsilon^4}\mathrm{d}\varepsilon. \tag{36}$$

Fig. 9 shows the dimensionless constriction resistance, that is the ratio of the constriction resistance and its permanent value $r_{\mathrm{cts}} = 8/3\pi^2 k r_0$, as a function of the dimensionless time $t^* = t/(r_0^2/a)$. It is difficult to define a "building-up time" of the constriction because the constriction resistance approaches its permanent value at a particularly slow pace: 0.90 at $t^* = 10$; 0.97 at $t^* = 100$; 0.99 at $t^* = 1000$. Here, we choose a little arbitrarily: $\tau_{\mathrm{ct}} \approx 10r_0^2/a$ (corresponding to $r_{\mathrm{ct}}^* = 0.9$).

5.1.2. Constriction impedance z_{ct}

The solution of problem (26) can be found by the use of the Laplace transformation

$$z_{\mathrm{ct}} = \frac{2}{\pi k r_0^2}\int_0^\infty \frac{J_1^2(\varepsilon)}{\varepsilon\sqrt{\varepsilon^2 + pr_0^2/a}}\mathrm{d}\varepsilon. \tag{37}$$

Fig. 10 shows the amplitude and the phase z_{ct}.

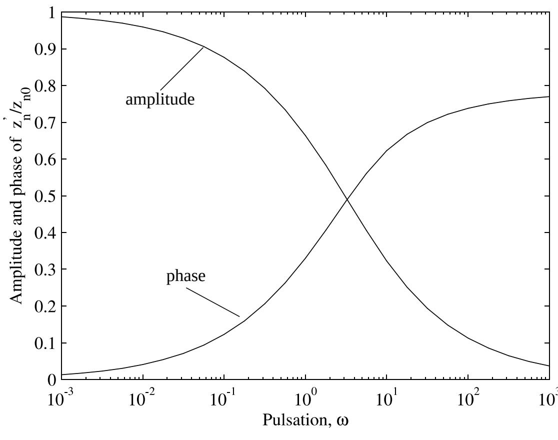

Fig. 10. Bode chart of the constriction impedance.

If we assume that the constriction impedance is established as soon as the amplitude of $z_{\rm ct} = 0.9 r_{\rm cts}$, we have

$$\frac{1}{p_{\rm ct}} \approx 60 r_0^2/a.$$

It is difficult to compare the two results because there is no direct relation between the time variable and the Laplace variable. For an order of magnitude, we use

$$\tau_{\rm ct} \approx \frac{1}{p_{\rm ct}}, \text{ i.e. } \tau_{\rm ct} \approx 60 r_0^2/a$$

that is relatively close to $\tau_{\rm ct} \approx 10 r_0^2/a$.

These results show that it is difficult to bring out a characteristic time of the constriction.

5.2. *Influence of the constriction building-up on the apparent resistance*

By comparison with the one-dimensional situation, an additional inflexion is present on the curves of Fig. 11. It is the consequence of the influence of the constriction building-up on the thermal field in the rod. Thus, the system presents three scales associated with characteristic times: contact time τ (supposing that τ_1 and τ_2 are of the same order); characteristic time of the rod $\tau_{\rm b} = l^2/a$; building-up time of the constriction $\tau_{\rm ct}$.

In Fig. 11, two examples are shown for $\tau_{\rm ct} \ll \tau_{\rm b}$ (characteristic times $\tau_{\rm ct} \approx 10 r_0^2/a$ are reported in Fig. 11): three states can be distinguished for thermal resistance $r_{\rm app}$ according to the contact period.

- *Large period*, $\tau \gg \tau_{\rm b} \gg \tau_{\rm ct}$: The constriction and the thermal field are quasi-steady. It corresponds to the situations of contact during phase τ_1 and noncontact during τ_2; the solution of the problem is simply the addition of two steady states.
- *Moderate period*, $\tau_{\rm b} \gg \tau \gg \tau_{\rm ct}$: The contact period is sufficiently short so that the thermal field in the rod has no time to evolve except in the constriction zone; the thermal field in the bar behaves then as in steady state, only the constriction resistance varies periodically with time.
- *Short period*, $\tau_{\rm b} \gg \tau_{\rm ct} \gg \tau$: The contact period is so small that, neither in the rod nor in the constriction zone, does the thermal field have time to evolve; the system resistance is then the same as in steady state with an average resistance at the interface equal to the inverse of the average conductance ($\bar{h} = (\tau_1 h_1 + \tau_2 h_2)/\tau$).

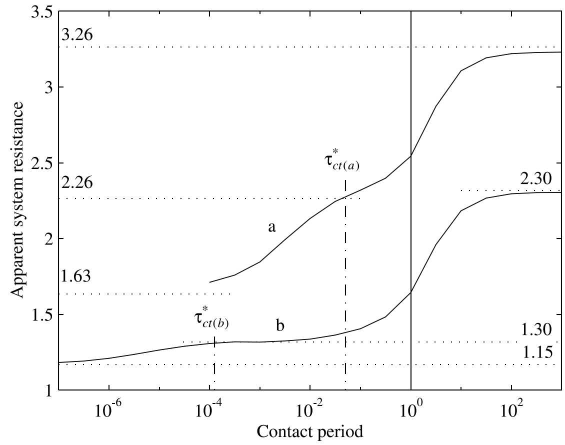

Fig. 11. Influence of the period on the apparent resistance. (a) $r_0/l = 0.04$, $R/l = 0.20$; (b) $r_0/l = 0.002$, $R/l = 0.020$.

From Figs. 8 and 11, we conclude therefore that in the domain $\tau \gg \tau_{\rm ct}$, the one-dimensional and two-dimensional models are identical for an equivalent boundary condition at the interface. On the contrary, for smaller periods, it is necessary to take into account the building-up of thermal constriction.

5.3. *Schemes for the three limiting states*

On the basis of the preceding analysis, we present three equivalent schemes of the apparent system resistance corresponding to three limiting states (Fig. 12):

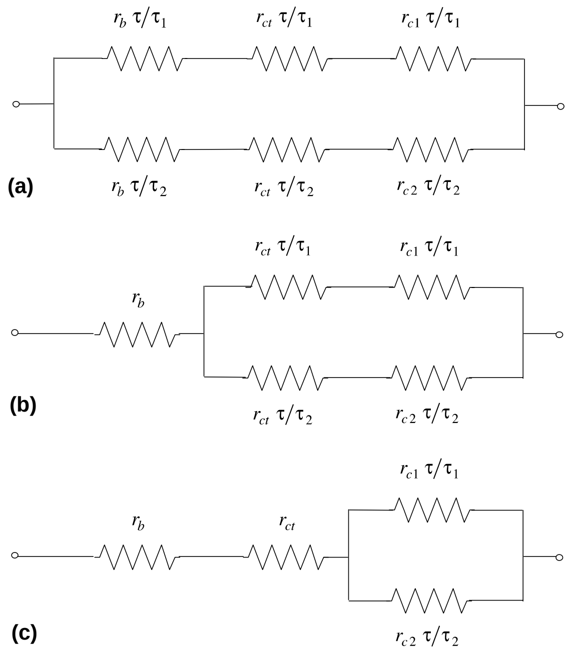

Fig. 12. Resistance schemes for three limiting states. (a) Large period state, $\tau \gg \tau_{\rm b} \gg \tau_{\rm ct}$; (b) moderate period state, $\tau_{\rm b} \gg \tau \gg \tau_{\rm ct}$; (c) short period state, $\tau_{\rm b} \gg \tau_{\rm ct} \gg \tau$.

- For large period state, we have two steady states, during the time τ_1 for the contact and the time τ_2 for the noncontact, from which:

  $$\Delta T = (r_b + r_{ct} + r_{c1})Q_1 = r_1 Q_1 \qquad \text{for } \tau_1,$$

  $$\Delta T = (r_b + r_{ct} + r_{c2})Q_2 = r_2 Q_2 \qquad \text{for } \tau_2.$$

  Calling on $\bar{Q}$ the flux average for the period, we have

  $\Delta T = r_{app}\bar{Q}$ with r_{app} the apparent resistance

  since

  $$\bar{Q} = \frac{Q_1\tau_1 + Q_2\tau_2}{\tau} = \frac{\Delta T}{r_1}\frac{\tau_1}{\tau} + \frac{\Delta T}{r_2}\frac{\tau_2}{\tau}$$

  and by identification

  $$r_{app} = \frac{1}{\frac{1}{r_1\tau/\tau_1} + \frac{1}{r_2\tau/\tau_2}},$$

  i.e. $1/(r_1\tau/\tau_1)$ in parallel with $1/(r_2\tau/\tau_2)$ (scheme a of Fig. 12).
- For moderate period state, the approach is identical considering the fact that the thermal field in the rod does not evolve except in the constriction zone, that is to say

  $$\Delta T = r_b\bar{Q} + (r_{ct} + r_{c1})Q_1 \quad \text{for } \tau_1,$$

  $$\Delta T = r_b\bar{Q} + (r_{ct} + r_{c2})Q_2 \quad \text{for } \tau_2,$$

  by identification, we obtain

  $$r_{app} = r_b + \frac{1}{\frac{1}{(r_{ct}+r_{c1})\tau/\tau_1} + \frac{1}{(r_{ct}+r_{c2})\tau/\tau_2}}$$

  (scheme b of Fig. 12).

- For short period state, the approach is identical considering the fact that neither in the rod nor in the constriction zone does the thermal field evolve, that is to say

  $$\Delta T = (r_b + r_{ct})\bar{Q} + r_{c1}Q_1 \qquad \text{for } \tau_1,$$

  $$\Delta T = (r_b + r_{ct})\bar{Q} + r_{c2}Q_2 \qquad \text{for } \tau_2,$$

  i.e.,

  $$r_{app} = (r_b + r_{ct}) + \frac{1}{\frac{1}{r_{c1}\tau/\tau_1} + \frac{1}{r_{c2}\tau/\tau_2}}$$

  (scheme c of Fig.12).

In the studied case where $h_1 \approx \infty$, $h_2 \approx 0$ and $\tau_1 = \tau_2$, the resistance networks give directly the apparent system resistance: large period state, $r_{app\infty} = 2(r_b + r_{ct})$; moderate period state, $r_{appm} = r_b + 2r_{ct}$; and short period state, $r_{app0} = r_b + r_{ct}$.

Table 1
Results obtained by resistance schemes

	r_b^*	r_{ct}^*	$r_{app\infty}^*$	r_{appm}^*	r_{app0}^*
Curve a	1.00	0.63	3.26	2.26	1.63
Curve b	1.00	0.15	2.30	1.30	1.15

(a): $r_0/l = 0.04$, $R/l = 0.20$; (b): $r_0/l = 0.002$, $R/l = 0.020$.

Table 1 shows the results of the schemes for two cases. In the calculation, we have

$$r_{ct} = A_0\frac{R^2}{r_0 l}r_b \text{ with}$$

$$A_0 = 0.848 - 1.093\frac{r_0}{R} + 0.245\left(\frac{r_0}{R}\right)^{3.75}$$

and $r_{c1,2} = \dfrac{1}{\pi R^2 h_{1,2}}$.

It is then immediate to verify that the three flats observed in both curves in Fig. 11 correspond to three asymptotic values obtained from the resistance schemes.

It should be noted that the quadrupole model and the preceding analyses remain valid for practical periodic micro-contact, where practical contact is modeled by a great number of cylindrical unit cells. The only difference is that the building-up time of constriction departs much more from the characteristic time of the rod as a consequence of an exceedingly small R/l value in practice.

6. Conclusion

A quadrupoles method has been developed to solve heat transfer through a macro-contact with thermal constriction, which is periodic in time and two-dimensional in space. The constriction is taken into account by the means of a constriction term present in the quadrupole matrix. The solution of the problem is based on Fourier developments of time periodic variables. The analytical approach is validated by a finite difference solution under the geometric condition that $l > R$.

The macro-contact study shows that the one-dimensional periodic model remains valid for long contact periods, however it is necessary to introduce the concept of "building-up" of the constriction to explain the thermal contact behavior for short and moderate periods. It is demonstrated that three characteristic times govern the problem: contact period τ, characteristic time of the rod τ_b and "building-up" time of constriction τ_{ct}. In the studied case where $\tau_{ct} \ll \tau_b$, three different asymptotic states can be observed according to the contact period compared with two other characteristic times. Simplified schemes of the apparent resistance are presented corresponding to the three limiting states.

The "building-up" time at 90% of the constriction resistance is analytically determined in a simple case: $\tau_{ct} \approx 10r_0^2/a$.

Appendix A. Fourier development of $h(t)$ (in the case of contact-noncontact)

The complex Fourier expansion of an absolutely integrable function $h(t)$ in the $[0, \tau]$ domain is defined by

$$h(t) \approx \sum h_n e^{i\omega_n t} \quad \text{with } \omega_n = \frac{2n\pi}{\tau},$$

where

$$h_n = \overline{h}_n = \frac{1}{\tau}\int_0^\tau h(t)e^{-i\omega_n t}dt.$$

Therefore the harmonics of the intermittent conductance

$$h(t) = \begin{cases} h_1, & 0 < t < \zeta\tau, \\ h_2, & \zeta\tau < t < \tau, \end{cases}$$

are

$$\begin{cases} h_0 = \zeta h_1 + (1-\zeta)h_2, \\ h_n = \frac{1-e^{-2n\pi\zeta i}}{2n\pi i}(h_1 - h_2), \quad n = 1,2,3,\ldots \end{cases}$$

References

[1] J.P. Bardon, Introduction à l'étude des résistances thermiques de contact, Rev. Gén. Therm. Fr. 11 (125) (1972) 429–447.
[2] C.V. Madhusudana, L.S. Fletcher, Contact heat transfer – the last decade, AIAA J. 24 (3) (1985) 510–523.
[3] J.P. Bardon, Heat transfer at solid–solid interface: basic phenomenons – recent works, Eurotherm 4 (1988) 40–74.
[4] L.S. Fletcher, Recent developments in contact conductance heat transfer, J. Heat Transfer – Trans. ASME 110 (1988) 1059–1071.
[5] C.V. Madhusudana, Thermal Contact Conductance, Springer, New York, 1996.
[6] J.R. Howard, A.E. Sutton, An analogue study of heat transfer through periodically contacting surfaces, Int. J. Heat Mass Transfer 13 (1970) 173–183.
[7] J.R. Howard, A.E. Sutton, The effect of thermal contact resistance on heat transfer between periodically contacting surfaces, J. Heat Transfer – Trans. ASME 95 (1973) 411–412.
[8] J.R. Reed, G. Mullineux, Quasi-steady state solution of periodically varying phenomena, Int. J. Heat Mass Transfer 16 (1973) 2007–2012.
[9] M.D. Mikhailov, Quasi-steady state temperature distribution in finite regions periodically-varying boundary conditions, Int. J. Heat Mass Transfer 17 (1974) 1475–1478.
[10] B. Vick, M.N. Özisik, Quasi-steady-state temperature distribution in periodically contacting finite regions, J. Heat Transfer – Trans. ASME 103 (1981) 739–744.
[11] A. Degiovanni, J.C. Batsale, Contact périodique en régime établi – Résistances thermiques apparentes limites pour des durées de contact nulles et infinies, Int. Commun. Heat Mass Transfer 23 (4) (1996) 579–587.
[12] G.P. Flach, M.N. Özisik, Periodic B-spline basis for quasi-steady periodic inverse heat conduction, Int. J. Heat Mass Transfer 30 (1987) 869–880.
[13] G.P. Flach, M.N. Özisik, Inverse heat conduction problem of periodically contacting surfaces, J. Heat Transfer – Trans. ASME 110 (1988) 821–829.
[14] J.R. Howard, An experimental study of heat transfer through periodically contacting surfaces, Int. J. Heat Mass Transfer 19 (1976) 367–372.
[15] W.M. Moses, R.R. Johnson, Experimental results for the quasi-steady heat transfer through periodically contacting surfaces, AIAA paper 87-1608, Honolulu, Hawaii, 1987.
[16] W.M. Moses, R.R. Johnson, Experimental study of the transient heat transfer across periodically contacting surfaces, AAIA J. Thermophys. Heat Transfer 2 (1988) 37–42.
[17] N.C. Dodd, W.M. Moses, Heat transfer across aluminum/stainless steel surfaces in periodic contact, AIAA paper 88-2646, Washington, DC, 1988.
[18] D. Couedel, F. Danes, J.P. Bardon, Experimental study and analysis of heat transfer in a valve-seat periodic contact in an internal combustion engine, ASME/HTD 182 (1991) 59–66.
[19] D. Maillet, S. Andre, J.C. Batsale, A. Degiovanni, Thermal Quadrupoles, Wiley, Chichester, UK, 2000.
[20] Y. Doin, A. Degiovanni, Contact thermique en régime variable et "milieu équivalent", CR Acad. Sci. Paris 297 (II) (1983) 397–400.
[21] A. Degiovanni, Conduction dans un mur multicouche avec sources: extension de la notion de quadripôle, Int. J. Heat Mass Transfer 31 (1988) 553–557.
[22] A. Degiovanni, A.S. Lamine, C. Moyne, Thermal contacts in transient states: a new model and two experiments, J. Thermophys. Heat Transfer 6 (1992) 356–363.
[23] H. Wang, Contribution à l'étude thermique des contacts périodiques solide–solide, Thèse de Doctorat d'Institut National Polytechnique de Lorraine, France, May 2000.
[24] Matlab user's guide, The Maths, Inc., South Natick, MA, USA.
[25] M. Abramowitz, I.A. Stegun (Eds.), Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, Dover, New York, 1972, p. 371.
[26] A. Degiovanni, G. Sinicki, A. Gery, M. Laurent, Un modèle de résistance thermique de contact en régime permanent, Rev. Gén. Therm. Fr. 267 (1984) 161–175.
[27] A. Degiovanni, Impédance de constriction, Rev. Gén. Therm. Fr. 34 (406) (1995) 623–624.

[28] H. Wang, C. Moyne, Contact thermique périodique avec constriction, in: Actes Congrès SFT: Thermique Aéronautique et Spatiale, vol. 5, Elsevier, Toulouse, France, 1997, pp. 731–736.

[29] G.E. Schneider, A.B. Strong, M.M. Yovanovich, Transient thermal response of two bodies communicating through a small circular contact area, Int. J. Heat Mass Transfer 20 (1977) 301–308.